Wavelength isolation sequence design

Jonathan Jedwab Mark Strange

23 August 2012 (revised 4 January 2013)

Abstract

A recent paper by Jedwab and Wodlinger renewed interest in a problem of multislit spectrometer design first proposed by Golay in 1951 but subsequently forgotten. It is shown that Golay's formulation of the problem in terms of 0/1 binary sequences is unduly restrictive. By relaxing the restrictions, infinitely many spectrometer designs satisfying all the original physical criteria can be found. Three constructions for such spectrometer designs are presented, involving Golomb rulers and variants. These constructions explain all nontrivial examples involving at most 26 slits.

Keywords binary sequence, Golay, Golomb ruler, multislit spectrometer, wavelength isolation

1 Introduction

In 1951, Golay [4] described a design for a multislit spectrometer that isolates desired radiation, having a single predetermined wavelength, from background radiation of all other wavelengths. The principle is to separate the incoming radiation into two streams and to pass each stream to its own detector; the detectors treat the background wavelengths equally, but treat the desired wavelength differentially. Both radiation streams pass through an entrance mask and exit mask comprising a pattern of open and closed slits, as in Figure 1. Background radiation that passes through an open slit of the entrance mask is diffracted either to the left or to the right, according to its wavelength. We may assume that desired radiation passes through an

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby BC V5A 1S6, Canada. Email jed@sfu.ca, mstrange@sfu.ca.

J. Jedwab is supported by an NSERC Discovery Grant. M. Strange was supported by an SFU Undergraduate Student Research Award.

open slit of the entrance mask without diffraction (by translating the exit mask if necessary).

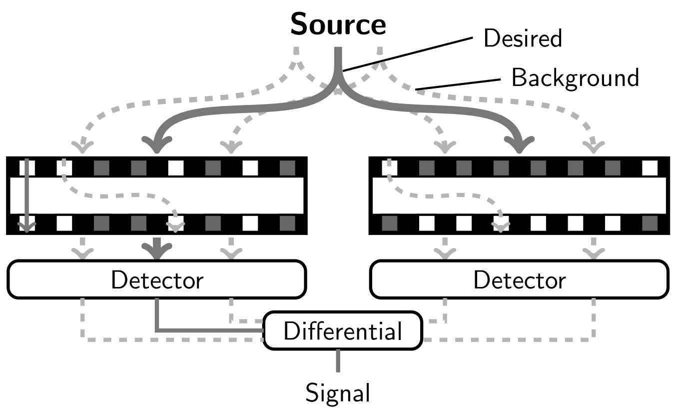

Figure 1: Golay's spectrometer design

The slit patterns are designed so that the energy of the radiation reaching each detector (after passing through an open slit of the entrance mask and an open slit of the exit mask) is equal for all background wavelengths, but different for the desired wavelength. The difference in total energy received by the two detectors is then entirely attributable to radiation of the desired wavelength. The design criterion is that the number of times an open slit of the exit mask lies at a displacement of v positions from an open slit of the entrance mask is the same in the two detectors exactly when $v \neq 0$. Figure 1 illustrates the case $v = 3$ (one time in both detectors) and the case $v = 0$ (four times in the left detector, never in the right detector).

We represent the pattern of n slits for each of the four masks by 0/1 binary sequences A, A', B, B' of length n (where 1s represents open slits and 0s represent closed slits), as in Figure 2. For real-valued sequences $X = (x_0, x_1, \ldots, x_{n-1})$ and $Y = (y_0, y_1, \ldots, y_{n-1})$ (we shall always index sequences starting from 0), define the *aperiodic crosscorrelation* of X and Y at shift $v \in \{0, 1, \ldots, n-1\}$ by $C_{X,Y}(v) = \sum_{i=0}^{n-1-v} x_i y_{i+v}$. We can then express the design criteria for the 0/1 sequence set (A, A', B, B') as

$$C_{A,A'}(v) = C_{B,B'}(v) \quad \text{for } 1 \leq v < n, \tag{1}$$

$$C_{A',A}(v) = C_{B',B}(v) \quad \text{for } 1 \leq v < n, \tag{2}$$

$$C_{A,A'}(0) > C_{B,B'}(0). \tag{3}$$

A	1	1	0	0	1	0	1	0		1	0	0	0	0	0	0	1	B
A'	1	1	0	0	1	0	1	0		0	1	1	1	1	1	1	0	B'

Figure 2: Binary sequences corresponding to spectrometer masks of Figure 1

The value $C_{A,A'}(0) - C_{B,B'}(0)$ is the *differential* of the desired wavelength as measured by the two detectors; we wish to find sequence sets (A, A', B, B') satisfying constraints (1) and (2) and (to allow sensitive measurements of the desired wavelength) having a large positive differential. Golay [4] proposed the simplification

$$A' = A \quad \text{and} \quad B' = \overline{B}, \tag{4}$$

where $\overline{B}$ represents the 0/1 complement of the sequence B. Jedwab and Wodlinger [7] called the subset (A, B) of a 0/1 sequence set (A, A', B, B') satisfying (1)–(4) a *wavelength isolation sequence pair* (WISP). They established the necessary condition that the sequence B of a WISP (A, B) be symmetric [7, Proposition 2], and produced nontrivial examples (in which the sequence A has at least two 1s) for lengths 3, 5, 7, 8, and 13 from perfect Golomb rulers (defined in Section 2 below) via two constructions [7, Theorem 9]. The examples at lengths 7 and 13 were unknown to Golay; in view of the apparent scarcity of examples, he proposed a different spectrometer design involving what are now known as Golay complementary pairs (see, for example, [5], [8], [3]).

This paper was motivated by the observation that Golay's proposed simplification $B' = \overline{B}$ is unduly restrictive: we need only satisfy constraints (1) and (2) with a large positive differential $C_{A,A'}(0) - C_{B,B'}(0)$. Writing $B = (b_i)$ and $B' = (b'_i)$, it turns out that relaxing the restriction (4) to

$$A' = A \quad \text{and } (b'_i, b_i) \neq (1, 1) \text{ for all } i \tag{5}$$

(that is, allowing the additional possibility that $(b'_i, b_i) = (0, 0)$ for one or more values of i) is sufficient to enable the existence of examples having an arbitrarily large differential. (The restriction (5) can be further relaxed, but it appears that the solution set is already sufficiently rich without needing to do so.) Rewriting the constraints (1)–(3) subject to (5), replacing B' by C, and writing $w(A)$ for the number of 1s in A, leads to the following definition.

Definition. *Let A, $B = (b_i)$, $C = (c_i)$ be 0/1 sequences of length n. The sequence set (A, B, C) forms a* wavelength isolation sequence triple (WIST) *of length n if $w(A) > 0$ and*

$$(b_i, c_i) \neq (1, 1) \text{ for all } i, \text{ and} \tag{6}$$

$$C_{A,A}(v) = C_{B,C}(v) = C_{C,B}(v) \quad \text{for } 1 \leq v < n. \tag{7}$$

The differential $C_{A,A}(0) - C_{B,C}(0)$ of a WIST (A, B, C) equals $w(A)$, since $C_{B,C}(0) = 0$ by (6). We seek WISTs having large differential.

2 Constructions

We shall give three constructions of WISTs, all of which can be applied to Golomb rulers; the second and third constructions can also be applied to more general inputs. A *Golomb ruler* R of length ℓ and order $d > 0$ is a set of d integers with least element 0 and greatest element ℓ, such that each integer in $\{1, 2, \ldots, \ell\}$ can be realised at most once as a difference of distinct elements of R. In the case that "at most once" can be replaced by "exactly once," the Golomb ruler is *perfect*. For example, $\{0, 4, 5, 7\}$ is a Golomb ruler of length 7, and $\{0, 1, 4, 6\}$ is a perfect Golomb ruler of length 6. A counting argument shows that $\ell \geq \binom{d}{2}$ for a Golomb ruler and $\ell = \binom{d}{2}$ for a perfect Golomb ruler.

Several infinite families of Golomb rulers, having length ℓ and order approximately $\sqrt{\ell}$, have been algebraically constructed (see [2] for a survey). In contrast, up to reversal and translation there are only four perfect Golomb rulers, one for each of the orders 1, 2, 3, 4 (a result attributed to Golomb in [1]).

The input R to our first WIST construction is an arbitrary Golomb ruler.

Theorem 1 (First WIST construction)**.** *Let R be a Golomb ruler of length ℓ and order d, and let D be the set of positive integers that are realised as a difference of distinct elements of R. For $1 \leq v \leq \ell$, let*

$$y_v = \begin{cases} 1 & \text{for } v \in D \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_i)$, B, C be the $0/1$ sequences of length $2\ell + 1$ given by

$$a_i = \begin{cases} 1 & \textit{for } i \in R \\ 0 & \textit{otherwise,} \end{cases}$$

$$\begin{array}{rcl} B & = & (0 \quad 0 \quad \ldots \quad 0 \;\; 1 \;\; 0 \quad \ldots \quad 0 \quad 0), \\ C & = & (y_\ell \quad y_{\ell-1} \quad \ldots \quad y_1 \;\; 0 \;\; y_1 \quad \ldots \quad y_{\ell-1} \;\; y_\ell). \end{array}$$

Then (A, B, C) is a length $2\ell + 1$ WIST with differential d.

Proof. It is immediate that $w(A) = d > 0$, condition (6) holds, and

$$C_{A,A}(v) = C_{B,C}(v) = C_{C,B}(v) = 0 \quad \text{for } \ell < v \leq 2\ell.$$

We therefore need show only that

$$C_{A,A}(v) = C_{B,C}(v) = C_{C,B}(v) \quad \text{for } 1 \leq v \leq \ell.$$

For v in this range, we have $C_{A,A}(v) = 1$ if and only if $v \in D$, which occurs if and only if $y_v = 1$. Therefore $C_{A,A}(v) = y_v$. On the other hand, by construction we have $C_{B,C}(v) = C_{C,B}(v) = y_v$. □

For example, let R be the Golomb ruler $\{0, 4, 5, 7\}$ of length 7 and order 4. Then $D = \{1, 2, 3, 4, 5, 7\}$, and the sequences

$$\begin{array}{l} A = (1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0), \\ B = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0), \\ C = (1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1) \end{array}$$

form a WIST of length 15 with differential 4.

Since there are Golomb rulers of arbitrarily large order, Theorem 1 constructs WISTs of arbitrarily large differential. The special case of Theorem 1 in which R is a perfect Golomb ruler, so that $D = \{1, 2, \ldots, \ell\}$, was given as the second construction of [7, Theorem 9]: the sequence pair (A, B) is then a WISP.

The set D of differences in our second WIST construction requires more structure than in Theorem 1, but the input R need not be a Golomb ruler.

Theorem 2 (Second WIST construction)**.** *Let R be a set of $d > 0$ integers with least element 0 and greatest element ℓ, and let D be the multiset of $\binom{d}{2}$ positive integers that are realised as a difference of distinct elements of R. Suppose that D can be written as the disjoint multiset union $\bigcup_{s \in S}(s + D_1)$*

for some set D_1, and some set S of integers with least element 0 *and greatest element $m > 0$ satisfying $s \in S \Leftrightarrow m - s \in S$. Let*

$$y_v = \begin{cases} 1 & \text{for } v \in D_1 \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le v \le \ell - m$, and let $A = (a_i)$, $B = (b_i)$, C be the $0/1$ sequences of length $2\ell - m + 1$ given by

$$a_i = \begin{cases} 1 & \text{for } i \in R \\ 0 & \text{otherwise,} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{for } i \in \ell - m + S \\ 0 & \text{otherwise,} \end{cases}$$

$$C = \ (y_{\ell-m} \quad y_{\ell-m-1} \ \ldots \ y_1 \ 0 \ \ldots \ 0 \ y_1 \ \ldots \ y_{\ell-m-1} \ y_{\ell-m}).$$

Then (A, B, C) is a length $2\ell - m + 1$ WIST *with differential d.*

Proof. The only part that is not immediate is

$$C_{A,A}(v) = C_{B,C}(v) = C_{C,B}(v) \quad \text{for } 1 \le v \le \ell.$$

For v in this range, $C_{A,A}(v)$ is the multiplicity of v in the multiset D, and therefore

$$\begin{aligned} C_{A,A}(v) &= \Big|\{s \in S : v \in s + D_1\}\Big| \\ &= \Big|\{s \in S : y_{v-s} = 1\}\Big| \\ &= \sum_{s \in S} y_{v-s} \end{aligned}$$

(where we define y_v to be 0 when v lies outside the range $1 \le v \le \ell - m$). But by construction we have $C_{C,B}(v) = \sum_{s \in S} y_{v-s}$, and since B is symmetric (because $s \in S \Leftrightarrow m - s \in S$) and C is symmetric we have $C_{B,C}(v) = C_{C,B}(v)$. $\square$

We note that, for a given multiset D in Theorem 2, there can be more than one choice for the sets D_1 and S. We also note that the condition $s \in S \Leftrightarrow m - s \in S$ of Theorem 2 always holds when $|S| = 2$.

A special case of Theorem 2 occurs when R is a Golomb ruler of length ℓ and order d, so that the multiset D is in fact a set. For example, let R once again be the Golomb ruler $\{0, 4, 5, 7\}$ of length 7 and order 4. Then

$D = \{1,2,3,4,5,7\}$ is the disjoint multiset union $\bigcup_{s \in S}(s + D_1)$, where $D_1 = \{1,2,5\}$ and $S = \{0,2\}$, and the sequences

$$
\begin{aligned}
A &= (1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0),\\
B &= (0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0),\\
C &= (1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1)
\end{aligned}
$$

form a WIST of length 13 with differential 4. This WIST has the same differential as that produced by Theorem 1, but is shorter and so can be implemented in a more compact spectrometer. In general, the WIST obtained by using a suitable Golomb ruler in Theorem 2 has the same differential but is shorter than that obtained by using the same Golomb ruler in Theorem 1. The special case of Theorem 2 in which R is a perfect Golomb ruler, so that we may take $D_1 = \{1\}$ and $S = \{0,1,\dots,\ell-1\}$, was given as the first construction of [7, Theorem 9]: the sequence pair (A, C) is then a WISP.

For an example with $|S| > 2$, let R be the Golomb ruler $\{0,2,7,10,11\}$ of length 11 and order 5. Then $D = \{1,2,3,4,5,7,8,9,10,11\}$, and we may take $D_1 = \{1,3\}$ and $S = \{0,1,4,7,8\}$ (which satisfies $s \in S \Leftrightarrow 8 - s \in S$) to give the sequences

$$
\begin{aligned}
A &= (1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0),\\
B &= (0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0),\\
C &= (1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1),
\end{aligned}
$$

which form a WIST of length 15 and differential 5.

For an example in which R is not a Golomb ruler, let $R = \{0,2,5,8,9\}$. Then $D = \{1,2,3,3,4,5,6,7,8,9\}$, and we may take $D_1 = \{1,2,3,6,7\}$ and $S = \{0,2\}$ to give the sequences

$$
\begin{aligned}
A &= (1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0),\\
B &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0),\\
C &= (1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1),
\end{aligned}
$$

which form a WIST of length 17 and differential 5.

Our third WIST construction is a modification of Theorem 2, in which some of the elements of the central block of 0s in the sequence C are now set to 1.

Theorem 3 (Third WIST construction). *Let R be a set of $d > 0$ integers with least element 0 and greatest element ℓ, and let D be the multiset of $\binom{d}{2}$ positive integers that are realised as a difference of distinct elements of R.*

Suppose that D can be written as the disjoint multiset union

$$\bigcup_{s\in S}(s+D_1) \quad \cup \bigcup_{\substack{(s,t)\in S\times T\\ s>t}} (s-t)$$

for some set D_1, some set S of integers with least element 0 and greatest element $m>0$ satisfying $s\in S \Leftrightarrow m-s\in S$, and some nonempty subset T of $\{0,1,\ldots,m\}$ disjoint from S and satisfying $t\in T \Leftrightarrow m-t\in T$. Let

$$y_v = \begin{cases} 1 & \text{for } v\in D_1 \\ 0 & \text{otherwise} \end{cases}$$

for $1\le v\le \ell-m$, and let $A=(a_i)$, $B=(b_i)$, $C=(c_i)$ be the $0/1$ sequences of length $2\ell-m+1$ given by

$$a_i = \begin{cases} 1 & \text{for } i\in R \\ 0 & \text{otherwise,} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{for } i\in \ell-m+S \\ 0 & \text{otherwise,} \end{cases}$$

$$c_i = \begin{cases} y_{\ell-m-i} & \text{for } 0\le i<\ell-m \\ y_{i-\ell} & \text{for } i>\ell \\ 1 & \text{for } i\in \ell-m+T \\ 0 & \text{otherwise.} \end{cases}$$

Then (A,B,C) is a length $2\ell-m+1$ WIST with differential d.

Proof. Since S and T are disjoint subsets of $\{0,1,\ldots,m\}$, we have $(b_i,c_i)\neq(1,1)$ for all i. The rest of the proof is similar to that of Theorem 2. It is sufficient to show that

$$C_{A,A}(v)=C_{B,C}(v)=C_{C,B}(v) \quad \text{for } 1\le v\le \ell.$$

For v in this range,

$$C_{A,A}(v)=\sum_{s\in S} y_{v-s}+\Big|\{(s,t)\in S\times T : s-t=v\}\Big|$$

(where we define y_v to be 0 when v lies outside the range $1\le v\le \ell-m$). But by construction we have

$$C_{C,B}(v)=\sum_{s\in S} y_{v-s}+\Big|\{(s,t)\in S\times T : (\ell-m+s)-(\ell-m+t)=v\}\Big|,$$

and since B is symmetric (because $s \in S \Leftrightarrow m - s \in S$) and C is symmetric (because $t \in T \Leftrightarrow m - t \in T$) we have $C_{B,C}(v) = C_{C,B}(v)$. $\square$

For example, let $R = \{0, 2, 3, 8, 10, 12, 13, 16, 17\}$. Then

$$D = \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 8, 8, 9, 9, 10, 10, 10, 11, 12, 13, 13, 14, 14, 15, 16, 17\},$$

and we may take $D_1 = \{1, 2\}$ and $S = \{0, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 15\}$ (satisfying $s \in S \Leftrightarrow 15 - s \in S$) and $T = \{5, 10\}$ (satisfying $t \in T \Leftrightarrow 15 - t \in T$) to give the sequences

$$\begin{aligned} A &= (1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0), \\ B &= (0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0), \\ C &= (1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1), \end{aligned}$$

which form a WIST of length 20 and differential 9.

Theorems 2 and 3 prompt the question: which sets R, and in particular which Golomb rulers, can be written as a disjoint multiset union of the required form?

3 Exhaustive search results

All WISTs (A, B, C) of length at most 26 were determined by an exhaustive search written in C, in which sequence elements are recursively fixed from the outermost positions inwards. The search algorithm takes the leftmost element of A to be 1, since an initial block of 0s in the leftmost positions of A can be removed and then appended to the rightmost positions. Likewise, it takes the leftmost element of either B or C to be 1. The algorithm takes the rightmost element of at least one of A, B, C to be 1, since otherwise the sequences could be truncated to a shorter WIST. It also takes $w(A) > 1$, in order to exclude trivial WISTs with $w(A) = 1$ such as those having $B = (0 \dots 0)$. We may transform such a nontrival WIST to an equivalent one by applying one or more of the following transformations:

1. Interchange sequences B and C.
2. Reverse the subsequence of A beginning at its leftmost 1 element and ending at its rightmost 1 element.
3. Reverse the subsequence of B beginning at its leftmost 1 element and ending at its rightmost 1 element, and do the same for C.

The search algorithm retains only one representative of each equivalence class of nontrivial WISTs determined by these transformations.

For example, there are six inequivalent nontrivial WISTs (A, B, C) of length 11, all of which are explained by the constructions of Section 2: Theorem 1 accounts for the WISTs

$$\begin{aligned} A &= (1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0), \\ B &= (0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0), \\ C &= (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1) \end{aligned}$$

and

$$\begin{aligned} A &= (1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0), \\ B &= (0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0), \\ C &= (1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1) \end{aligned}$$

and

$$\begin{aligned} A &= (1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0), \\ B &= (0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0), \\ C &= (1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1), \end{aligned}$$

Theorem 2 accounts for the WIST

$$\begin{aligned} A &= (1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0), \\ B &= (0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0), \\ C &= (1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1), \end{aligned}$$

and Theorem 3 accounts for the WISTs

$$\begin{aligned} A &= (1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0), \\ B &= (0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0), \\ C &= (1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1) \end{aligned}$$

and

$$\begin{aligned} A &= (1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0), \\ B &= (0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0), \\ C &= (1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1). \end{aligned}$$

A complete listing of the inequivalent nontrivial WISTs of length at most 26 and their corresponding aperiodic crosscorrelations is contained in [6]. A summary is given in Table 1, which shows the total number of inequivalent nontrivial WISTs of length at most 26, and the maximum differential occurring at each length. The counts in the table demonstrate that the three constructions of Section 2 account for the existence of every nontrivial WIST of length at most 26. Is there a longer nontrivial WIST that cannot be explained by these constructions?

We note that the constructed sequences B and C of Theorems 1, 2 and 3 are always symmetric, which implies by the search result above that the sequences B and C of each nontrivial WIST (A, B, C) of length at most 26 are both symmetric. Does the same hold at all lengths? (We know that symmetry of B is forced for nontrivial WISTs $(A, B, \overline{B})$ at all lengths [7, Proposition 2].)

4 Conclusion

We have shown for the first time how to construct infinitely many solutions to Golay's spectrometer design problem of 1951, by relaxing Golay's formulation of the corresponding 0/1 sequence design problem to require a wavelength isolation sequence triple (WIST) rather than a wavelength isolation sequence pair. These solutions allow an arbitarily large measured differential of the desired wavelength at the detectors. The WIST constructions given in Theorems 1, 2 and 3, involving Golomb rulers and variants, account for all nontrivial WISTs of length at most 26.

We conclude by repeating some questions raised in the paper.

1. Must the sequences B and C of a nontrivial WIST (A, B, C) be symmetric?

2. Is there a nontrivial WIST of length greater than 26 that cannot be explained by Theorems 1, 2 and 3?

3. Which sets R, and in particular which Golomb rulers, can be written as a disjoint multiset union of the form required by Theorem 2 or 3?

Acknowledgements

We are grateful to Jane Wodlinger for many helpful discussions, and to Karen Yeats for providing us with the first example of a WIST (A, B, C) having $C \neq \overline{B}$.

References

[1] A. Dimitromanolakis. Analysis of the Golomb ruler and the Sidon set problems, and determination of large, near-optimal Golomb rulers. Diploma thesis, Technical University of Crete, 2002.

[2] K. Drakakis. A review of the available construction methods for Golomb rulers. *Adv. Math. Commun.*, **3**:235–250, 2009.

[3] F. Fiedler, J. Jedwab, and M.G. Parker. A multi-dimensional approach to the construction and enumeration of Golay complementary sequences. *J. Combin. Theory (A)*, **115**:753–776, 2008.

[4] M.J.E. Golay. Static multislit spectrometry and its application to the panoramic display of infrared spectra. *J. Opt. Soc. Amer.*, **41**:468–472, 1951.

[5] M.J.E. Golay. Complementary series. *IRE Trans. Inform. Theory*, **IT-7**:82–87, 1961.

[6] J. Jedwab and M. Strange. Wavelength isolation sequence design: supplementary data. Technical report, Simon Fraser University, January 2013. `<http://summit.sfu.ca/item/10999>`.

[7] J. Jedwab and J. Wodlinger. Wavelength isolation sequence pairs. In T. Helleseth and J. Jedwab, editors, *Sequences and Their Applications — Proceedings of SETA 2012*, volume 7280 of *Lecture Notes in Computer Science*, pages 126–135. Springer-Verlag, 2012.

[8] R.J. Turyn. Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings. *J. Combin. Theory (A)*, **16**:313–333, 1974.

Length	# explained by Theorem 1	# explained by Theorem 2	# explained by Theorem 3	# found by exhaustive search	Maximum differential
2	0	0	0	0	—
3	1	0	0	1	2
4	0	0	0	0	—
5	1	1	0	2	3
6	0	0	0	0	—
7	2	0	0	2	3
8	0	1	0	1	4
9	2	2	0	4	4
10	0	2	0	2	4
11	3	1	2	6	5
12	0	1	1	2	4
13	4	4	3	11	5
14	0	2	1	3	4
15	7	2	2	11	5
16	0	3	0	3	4
17	8	6	1	15	5
18	0	2	1	3	4
19	14	8	8	30	6
20	0	2	2	4	9
21	13	6	12	31	9
22	0	5	1	6	4
23	23	6	3	32	5
24	0	2	4	6	9
25	30	15	9	54	9
26	0	2	1	3	4

Table 1: Number of inequivalent nontrivial WISTs of length at most 26 explained by the constructions of Section 2, total number found by exhaustive search, and maximum differential at each length